First Fundamental Theorem of Invariant Theory for covariants of classical groups.

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Abstract. Let $U(G)$ be a maximal unipotent subgroup of one of classical groups $G = GL(V), O(V), Sp(V)$. Let W be a direct sum of copies of V and its dual V^*. For the natural action $U(G) : W$, we describe a minimal system of homogeneous generators for the algebra of $U(G)$-invariant regular functions on W. For $G = GL(V)$, we also describe the syzygies among these generators in some particular cases.

1 Main theorem.

Let V be a finite-dimensional vector space over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $H \subseteq GL(V)$ be an algebraic subgroup. For any $l \in \mathbf{N}$, we denote by lV the direct sum of l copies of V; similarly, we define mV^* for any $m \in \mathbf{N}$. Consider the natural action of H on $W = lV \oplus mV^*$ and assume that the algebra $\mathbf{k}[W]^H$ of invariants is finitely generated for any l, m. Then First Fundamental Theorem of Invariant Theory of H refers to a description of a minimal system of homogeneous generators of $\mathbf{k}[W]^H$ for all l, m.

Such a description exists when H is *classical*, i.e., H is one of groups $GL(V)$, $SL(V)$, $O(V)$, $SO(V)$, $Sp(V)$ (see e.g. [PV, §9]).

Let now G be one of the groups $GL(V), O(V), Sp(V)$; let $U(G)$ be a maximal unipotent subgroup of G. By [PV, Theorem 3.13], the algebra $\mathbf{k}[W]^{U(G)}$ is finitely generated. Also the invariants of $U(G)$ are linear combinations of highest vectors of irreducible factors for G-module $\mathbf{k}[W]$. So the $U(G)$-invariants are the G-covariants and the First Fundamental Theorem for covariants of G means that for the invariants of $H = U(G)$.

Using a result (and some ideas) of [Ho], we prove in this paper First Fundamental Theorem for covariants of each of the above classical G.

Note that for $G = Sp(V), O(V)$, V and V^* are isomorphic as G-modules, hence, as $U(G)$-modules. Therefore we may assume $m = 0$ in these cases.

Elements of $\wedge^k V^* \subseteq \otimes^k V^* \subseteq \mathbf{k}[kV]$, $k \leq \dim V$ are said to be *multilinear anisymmetric* functions as well as their analogs in $\mathbf{k}[kV^*]$.

Theorem 1 *The algebra $\mathbf{k}[W]^{U(G)}$ is generated by the subalgebra $\mathbf{k}[W]^G$ and multilinear antisymmetric invariants. Moreover, a set $\mathcal{M}$ described below is a minimal system of homogeneous generators of $\mathbf{k}[W]^{U(G)}$.*

We describe a minimal system $\mathcal{M}$ of homogeneous generators of $\mathbf{k}[W]^{U(G)}$ in a coordinate form. Set $n = \dim V$, choose a basis of V and denote by $\overline{V}$ the corresponding $n \times l$-matrix of coordinates on lV. Similarly, denote by $\overline{V^*}$ the $m \times n$-matrix of coordinates on mV^*, in the dual basis of V^*. A minor of order k of a matrix is said to be *left*, if it involves the first k columns. Analogeously, we call it *lower*, if it involves the last k rows.

A) Let $G = GL(V)$ and define $U(GL) = U(GL(V))$ to be the subgroup of the strictly upper triangular matrices, in the above basis. Then $\mathcal{M}$ is:
• the matrix elements of the product $\overline{V^*V}$
• the lower minor determinants of order k of $\overline{V}$, $k = 1, \cdots, \min\{l, n\}$.
• the left minor determinants of order p of $\overline{V^*}$, $p = 1, \cdots, \min\{m, n\}$.

Let $T(GL)$ be the diagonal matrices in the above basis. Then $T(GL)$ is a maximal torus of G normalizing $U(GL)$. The pair $T(GL), U(GL)$ defines a system of simple roots of $T(GL)$. Here and in what follows, we use the enumeration of simple roots of simple groups as in [OV] and denote by $\varphi_1, \cdots, \varphi_n$ the fundamental weights. The torus $T(GL)$ acts on $\mathbf{k}[W]^{U(GL)}$ and the elements of $\mathcal{M}$ are weight vectors of $T(GL)$. The set of their degrees and weights is (for $l, m \geq n$):

$$(2,0), (1,\varphi_1), (2,\varphi_2), \cdots, (n,\varphi_n),$$
$$(1, \varphi_{n-1} - \varphi_n), (2, \varphi_{n-2} - \varphi_n), \cdots, (n-1, \varphi_1 - \varphi_n), (n, -\varphi_n).$$

Furthermore, let Q be a bilinear symmetric (antisymmetric) form having in the above basis a matrix with ± 1 on the secondary diagonal and with zero entries outside it. Define $G = O(V)$ ($G = Sp(V)$) to be the stabilizer of this form. Then $U(G) = G \cap U(GL)$ is a maximal unipotent subgroup in G. Moreover, set $T(O) = T(GL) \cap SO(V)$, $T(Sp) = T(GL) \cap Sp(V)$. Then $T(G)$ is a maximal torus of G of rank $r = [\frac{n}{2}]$. Denote by $\varphi_1, \cdots, \varphi_r$ the fundamental weights of $T(G)$ with respect to $U(G)$. For $x \in W = lV$, denote by v_i the projections of x on the i-th V-factor, $i = 1, \cdots, l$.

B) Let $n = 2r+1, G = O(V)$. Then $\mathcal{M}$ is:
• $Q(v_i, v_j), 1 \leq i \leq j \leq l$,
• the lower minor determinants of order k of $\overline{V}$, $k = 1, \cdots, \min\{l, n\}$,

The set of degrees and weights of the above generators is (for $l \geq n$):

$$(2,0), (1,\varphi_1), \cdots, (r-1, \varphi_{r-1}), (r, 2\varphi_r), (r+1, 2\varphi_r), \cdots, (n-1, \varphi_1), (n, 0).$$

C) Let $G = Sp(V)$. Then $\mathcal{M}$ is:
• $Q(v_i, v_j), 1 \leq i < j \leq l$,
• the lower minor determinants of order k of $\overline{V}$, $k = 1, \cdots, \min\{l, r\}$.

The set of degrees and weights of the above generators is (for $l \geq r$):

$$(2, 0), (1, \varphi_1), (2, \varphi_2), \cdots, (r, \varphi_r).$$

Note that the lower minor determinants of order k of $\overline{V}$ with $k > r$ are $U(Sp)$-invariant, too. It is not hard to check that these can be expressed in the above generators.

D) Let $n = 2r, G = O(V)$. Then $\mathcal{M}$ is:
• $Q(v_i, v_j), 1 \leq i \leq j \leq l$,
• the lower minor determinants of order k of $\overline{V}$, $k = 1, \cdots, \min\{l, n\}$,
• for $l \geq r$, the minor determinants of order r, involving the r-th row and the last $r-1$ rows of $\overline{V}$.

The set of degrees and weights of the above generators is (for $l \geq n$):

$$(2, 0), (1, \varphi_1), \cdots, (r-2, \varphi_{r-2}), (r-1, \varphi_{r-1} + \varphi_r), (r, 2\varphi_{r-1}), (r, 2\varphi_r),$$
$$(r+1, \varphi_{r-1} + \varphi_r), \cdots, (n-1, \varphi_1), (n, 0).$$

2 Proof of Theorem 1.

First we state a result of [Ho] that is a starting point of our proof. We keep the notation of loc.cit. but consider a slightly more general setting.

Let W be a finite dimensional $\mathbf{k}$-vector space. Denote by

$$\mathfrak{gr} = \mathfrak{gr}_{(2,0)} \oplus \mathfrak{gr}_{(1,1)} \oplus \mathfrak{gr}_{(0,2)} \subseteq \mathrm{End}\mathbf{k}[W]$$

the linear subspace of differential operators with the prescribed by the index degree and order. Namely, $\mathfrak{gr}_{(2,0)}$ are the homogeneous regular functions on W of degree 2 acting on $\mathbf{k}[W]$ by multiplication; $\mathfrak{gr}_{(0,2)}$ are the constant coefficients differential operators of order 2; $\mathfrak{gr}_{(1,1)}$ is nothing but the Lie algebra $\mathfrak{gl}(W)$.

Clearly, $\mathfrak{gr}$ is a Lie subalgebra in $\mathrm{End}\mathbf{k}[W]$, and moreover, $\mathfrak{gr}$ is isomorphic to $\mathfrak{sp}(W \oplus W^*)$, with respect to the natural symplectic form on $W \oplus W^*$.

Assume now that $G \subseteq GL(W)$ is a reductive subgroup. Then G acts on $\mathfrak{gr}$; consider the invariants:

$$\Gamma' = \mathfrak{gr}^G, {\Gamma'}_{(2,0)} = \mathfrak{gr}^G_{(2,0)}, {\Gamma'}_{(1,1)} = \mathfrak{gr}^G_{(1,1)}, {\Gamma'}_{(0,2)} = \mathfrak{gr}^G_{(0,2)}.$$

Clearly, $\Gamma' = {\Gamma'}_{(2,0)} \oplus {\Gamma'}_{(1,1)} \oplus {\Gamma'}_{(0,2)}$ is also a Lie subalgebra in $\mathrm{End}\mathbf{k}[W]$.

Let $\mathbf{k}[W] = \bigoplus_{k=1}^{\infty} I_k$ be the decomposition of G-module $\mathbf{k}[W]$ into isotypic components. Let I be one of I_k. Clearly, I is stable under the action of Γ'.

Theorem 2 ([Ho, Theorem 8]) *Assume that the algebra* $\mathbf{k}[W \oplus W^*]^G$ *of invariants is generated by elements of degree 2. Then* I *is an irreducible joint* (G, Γ')*-module.*

By the First Fundamental Theorem for the classical groups, the assumption of Theorem 2 holds for the pairs (G, W) from section 1. For these particular cases the above Theorem is (a part of) Theorem 8 of [Ho]. However, one can see that the proof in loc.cit. works whenever the assumption of Theorem 2 holds.

Note that for the classical (G, W) we have: $\Gamma'_{(1,1)} = \mathfrak{gl}_l \oplus \mathfrak{gl}_m$,

$$\Gamma' \cong \mathfrak{gl}_{l+m}, \text{ if } G = GL(V), \Gamma' \cong \mathfrak{sp}_{2l}, \text{ if } G = O(V), \Gamma' \cong \mathfrak{o}_{2l}, \text{ if } G = Sp(V).$$

We now show that Theorem 2 reduces Theorem 1 to a more simple statement. The below reasoning is an analog of that from the proof of Theorem 9 in loc.cit.

Clearly, I is a homogeneous submodule of $\mathbf{k}[W]$; denote by I^{min} the subspace of the elements of I of minimal degree. Let $A \subseteq \mathbf{k}[W]^U$ be the subalgebra generated by $\mathcal{M}$. Let $Z \subseteq \mathbf{k}[W]$ be the G-submodule generated by A. Then Theorem 1 can be reformulated as follows: $Z = \mathbf{k}[W]$. Assume that $X = Z \cap I^{min}$ is nonzero.

Since the system $\mathcal{M}$ of generators of A is symmetric with respect to permutations of isomorphic G-factors of W, A is $GL_l \times GL_m$-stable, i.e., $\Gamma'_{(1,1)}$-stable. Hence, Z and X are stable with respect to both G and $\Gamma'_{(1,1)}$.

Let $R, R_{(2,0)}$ etc. be the subalgebras in $\mathrm{End}\mathbf{k}[W]$ generated by $\Gamma', {\Gamma'}_{(2,0)}$ etc. Consider R as a representation of the universal enveloping algebra of Γ'. Using the PBW theorem, we obtain

$$R = R_{(2,0)} R_{(1,1)} R_{(0,2)}. \tag{1}$$

Differentiating a polynomial, we decrease its degree; hence, ${\Gamma'}_{(0,2)} I^{min} = 0$. Therefore $R_{(0,2)} X = X$. Moreover, since X is $\Gamma'_{(1,1)}$-stable, we have by (1): $RX = R_{(2,0)} X = \mathbf{k}[W]^G X$. On the other hand, RX is a non-zero joint (G, Γ')-submodule of I. By Theorem 2, $I = RX = \mathbf{k}[W]^G X \subseteq Z$.

Thus to prove Theorem 1, we need to check for any isotypic component I:

$$A \cap I^{min} \neq \{0\}. \tag{2}$$

Note that it is sufficient to prove Theorem 1 with $l, m \geq n$, in the case $G = GL(V)$, and with $l \geq n, m = 0$, in the case $G = O(V), Sp(V)$.

Denote by G^0 the connected component of the unity of G; $GL(V)$ and $Sp(V)$ are connected, but for $G = O(V)$, $G^0 = SO(V)$. Recall that the irreducible finite dimensional G^0-modules are in one-to-one correspondence with their highest weights with respect to $U(G)$ and $T(G)$. Denote by P the set of highest weights of irreducible factors for G^0-module $\mathbf{k}[W]$. For any graded algebra B and $t \in \mathbf{N}$,

we denote by B_t the subspace of the elements of degree t. For any $\chi \in P$ we set:

$R(\chi)$ is the irreducible representation of G^0 with highest weight χ

I_χ is the $R(\chi)$-isotypic component of G^0-module $\mathbf{k}[W]$

$m(\chi) = \min\{t | \mathbf{k}[W]_t \cap I_\chi \neq 0\}$.

$n(\chi) = \min\{t | A_t \cap I_\chi \neq 0\}$.

By definition, $n(\chi) \geq m(\chi)$. For $G = GL(V), Sp(V)$ the condition (2) is equivalent to $n(\chi) = m(\chi)$ for any $\chi \in P$.

Lemma 1 *For any $\chi \in P, c \in \mathbf{N}$ we have: $n(c\chi) = cn(\chi)$.*

Denote by $\mathfrak{t}$ the Lie algebra of $T(G)$. Let $\mathcal{C} \subseteq \mathfrak{t}^*$ be the Weyl chamber corresponding to $U(G)$. Consider the set

$$\Delta = \{\frac{\chi^*}{t} | I(\chi) \cap \mathbf{k}[W]_t \neq 0\} \subseteq \mathcal{C},$$

where χ^* denotes the highest weight of the G^0-module dual to that with highest weight χ. By [Br87], if $\mathbf{k}$ is the field $\mathbf{C}$ of complex numbers, then Δ is the set of rational points in the momentum polytope for the action of the maximal compact subgroup $K \subseteq G^0$ on the projective space $\mathbf{P}(W)$. Further, we set:

$$\widetilde{\Delta} = \{\frac{\chi^*}{t} | I(\chi) \cap Z_t \neq 0\} \subseteq \Delta.$$

Let now $\Phi \subseteq \mathfrak{t}^*$ be the convex hull over the rational numbers of the weights for the action $T(G) : W$.

Lemma 2 *$\widetilde{\Delta} \supseteq \Phi \cap \mathcal{C}$.*

By definition, we have: $\Delta \subseteq \Phi \cap \mathcal{C}$. Therefore $\Delta = \widetilde{\Delta} = \Phi \cap \mathcal{C}$ [1].

Suppose that $\mathbf{k}[W]^{U(G)}$ contains an element of degree t and weight χ. Then by definition, $\frac{\chi^*}{t} \in \Delta$. Hence, the equality $\Delta = \widetilde{\Delta}$ implies that for some $c \in \mathbf{N}$ there exists an element of A of degree ct and weight $c\chi$. Thus $ct \geq n(c\chi) = cn(\chi)$ and $t \geq n(\chi)$. In other words, $m(\chi) \geq n(\chi)$, hence $m(\chi) = n(\chi)$. This completes (modulo Lemmas 1 and 2) the proof of Theorem for $G = GL(V), Sp(V)$.

Let G be $O(V)$; to prove Theorem, we apply induction on $n = \dim V$.

For $n = 2$, $U(O)$ is trivial and one can see $A = \mathbf{k}[W]$.

For $n = 3$, $(SO_3, \mathbf{k}^3) \cong (SL_2, S^2\mathbf{k}^2)$. Since the stabilizer of a point on the dense orbit for the action $SL_2 : \mathbf{k}^2$ is a maximal unipotent subgroup in SL_2, we obtain an isomorphism:

$$\mathbf{k}[\mathbf{k}^2 + l\mathbf{k}^3]^{SL_2} \cong \mathbf{k}[W]^{U(O)}.$$

[1] For $\mathbf{k} = \mathbf{C}$, one can directly prove for the moment polytope $\Delta \otimes \mathbf{R} = (\Phi \otimes \mathbf{R}) \cap \mathcal{C}$.

Lemma 3 *There exists an isomorphism* $\mathbf{k}[\mathbf{k}^2 + l\mathbf{k}^3]^{SL_2} \cong \mathbf{k}[(l+1)\mathbf{k}^3]^{SO_3}/(d)$, *where* $d = Q(v_{l+1}, v_{l+1})$.

Proof: Consider the morphism

$$\varphi : \mathbf{k}^2 + l\mathbf{k}^3 \to (l+1)\mathbf{k}^3, \varphi(e, Q_1, \cdots, Q_l) = (Q_1, \cdots, Q_l, e^2).$$

Clearly, φ is SL_2-equivariant; moreover, φ is the quotient map with respect to the center of SL_2. Furthermore, the image of φ is the zero level of d. This completes the proof.□

Using Lemma 3 and the well-known description of $\mathbf{k}[(l+1)\mathbf{k}^3]^{SO_3}$, one easily deduces the Theorem for $n = 3$.

The step of induction. Assume that Theorem is proven for $n-2$. We apply now the Theorem of local structure of Brion-Luna-Vust ([BLV]) to get a local version of the assertion of Theorem.

Denote by $x_i^j = \overline{V}_i^j$ the i-th coordinate of v_j. Set $f = x_n^1 \in \mathbf{k}[W]^U$, $W_f = \{x \in W | f(x) \neq 0\}$. Define a mapping:

$$\psi_f : W_f \to \mathfrak{o}(V)^*, \psi(x)(\xi) = \frac{(\xi f)(x)}{f(x)}.$$

Denote by P_f the stabilizer in $SO(V)$ of the line $\langle f \rangle$. Clearly, P_f is a parabolic subgroup in $SO(V)$ containing $U(O)$ and ψ_f is P_f-equivariant.

Furthermore, we denote by e_i^j the i-th element of the above basis in the j-th copy of V, $x = e_n^1$, $\Sigma = \psi_f^{-1}(\psi_f(x))$. Denote by L the stabilizer of $\psi_f(x)$ in P_f. By [BLV], L is a Levi subgroup of P_f and the natural morphism

$$P_f *_L \Sigma \to W_f, (p, \sigma) \to p\sigma$$

is a P_f-equivariant isomorphism. Therefore we have:

$$\mathbf{k}[W]_f^{U(O)} \cong \mathbf{k}[W_f]^{U(O)} \cong \mathbf{k}[P_f *_L \Sigma]^{U(O)}.$$

Also, $P_f = U(O)L$. Hence,

$$\mathbf{k}[P_f *_L \Sigma]^{U(O)} \cong \mathbf{k}[\Sigma]^{U(O)\cap L} = \mathbf{k}[\Sigma]^{U(L)},$$

where $U(L)$ is a maximal unipotent subgroup in L. Calculating, we have:

$$(L, \Sigma) \cong (SO_2 \times SO_{n-2}, \langle e_1^1, e_n^1 \rangle_f \times (l-1)V).$$

In other words,

$$\mathbf{k}[W]_{x_n^1}^{U(O)} \cong \mathbf{k}[x_1^1, x_n^1, x_1^2, x_n^2, \cdots, x_1^l, x_n^l]_{x_n^1} \otimes \mathbf{k}[(l-1)\mathbf{k}^{n-2}]^{U(O_{n-2})}.$$

The induction hypothesis yields the generators of $\mathbf{k}[(l-1)\mathbf{k}^{n-2}]^{U(O_{n-2})}$. Restricting the elements of $\mathcal{M}$ to Σ, one can easily deduce:

$$\mathbf{k}[W]_{x_n^1}^{U(O)} = A_{x_n^1}. \tag{3}$$

We return to our consideration of the isotypic components of $O(V) : \mathbf{k}[W]$. Consider an irreducible representation ρ of $O(V)$ and its restriction ρ' to $SO(V)$. Here two cases occur:
• either ρ' is also irreducible, $\rho' = R(\chi)$ for some $\chi \in P$
• or else $n = 2r$, $\rho' = R(\chi) + R(\tau(\chi))$, where τ is the automorphism of the system of simple roots of $O(V)$ interchanging the $r-1$-th and the r-th roots.

The latter case is more simple: elements of minimal degree in the ρ-isotypic component are the elements of minimal degree in both $I(\chi)$ and $I(\tau(\chi))$ (clearly, $n(\chi) = n(\tau(\chi))$ and $m(\chi) = m(\tau(\chi))$). Hence, the above equality $n(\chi) = m(\chi)$ implies the assertion for such an isotypic component.

Now consider the former case. Here for any $\rho' = R(\chi)$ there exist two possibilities for ρ: $R(\chi_+)$ and $R(\chi_-) = R(\chi_+) \otimes \det$, where det is the unique nontrivial character of $O(V)$. Moreover, we define explicitly $R(\chi_+)$ and $R(\chi_-)$ as follows. Let $\theta \in O(V) \setminus SO(V)$ be an element normalizing $T(O)$ as follows. For n odd, $\theta = -Id$. For n even, θ is the operator interchanging the r-th and the $r+1$-th elements of the above basis and acting trivially on the other basis elements. Note that in both cases $\theta(\chi) = \chi$ for any χ, if n is odd and for all χ such that $\tau(\chi) = \chi$, if n is even. Now we define $R(\chi_\pm)$ by the condition:

$$R(\chi_\pm)(\theta)(u_\chi) = \pm u_\chi$$

for the highest vector u_χ of $T(O)$ and $U(O)$ in $R(\chi)$. For instance, if n is even, $k \le r-2$, minor determinants of order k of $\overline{V}$ generate $R(\varphi_{k+})$ and minor determinants of order $n-k$ generate $R(\varphi_{k-})$. Moreover, multiplying two highest vectors of $\mathbf{k}[W]$, we add their weights and multiply as usual their $\pm$ subscripts. Thus we control the structure of the $O(V)$-module Z.

Define $m(\chi_\pm), n(\chi_\pm)$ as above. Then the condition (2) is equivalent to the equality $m(\chi_\pm) = n(\chi_\pm)$ for any $\chi \in P$ (τ-invariant for n even). For any $\chi = \sum_{i=1}^{q} k_i\varphi_i, k_q > 0$, set $t = r-1$, if $q = r, n = 2r$ and $t = q$ otherwise. Then we have:

$$\min\{n(\chi_+), n(\chi_-)\} = n(\chi), |n(\chi_+) - n(\chi_-)| = n - 2t. \tag{4}$$

Let g be a highest vector of $\mathbf{k}[W]$ generating $R(\chi_-)$. Then by (3), for some even j we have:$(x_n^1)^j g \in A$. Since $(x_n^1)^j g$ generates $R((\chi + j\varphi_1)_-)$, we have: $\deg g + j \ge n((\chi + j\varphi_1)_-)$. Clearly, $n(\chi + j\varphi_1) = n(\chi) + j$ (see formulae (5),(6) below). Hence, (4) yields $n((\chi + j\varphi_1)_-) = n(\chi_-) + j$. Thus we have $\deg g \ge n(\chi_-)$ implying $m(\chi_-) = n(\chi_-)$. The same is true for χ_+. This completes the proof of Theorem for $G = O(V)$.□

Thus we reduced Theorem 1 to Lemma 1 and Lemma 2. Both are properties of degrees and weights of the given generators, and we consider case by case.

3 Proof of Lemmas 1 and 2.

Proof of Lemma 1.

Recall that $n(\chi)$ is the minimum of degree of the monomials in the elements of $\mathcal{M}$ having weight χ. Clearly, we should not involve the G-invariants in a monomial of minimal degree. Then for $G = Sp(V), O(V)$ we have no much choice for such a monomial and we can write down formulae for $n(\chi)$ as follows. Let $\chi = k_1\varphi_1 + \cdots + k_r\varphi_r$.

For $G = Sp(V)$, we have: $n(\chi) = k_1 + 2k_2 + \cdots + rk_r$.

For $G = O(V), n = 2r + 1$, k_r is even for $\chi \in P$, and we have:

$$n(\chi) = k_1 + 2k_2 + \cdots + (r-1)k_{r-1} + r\frac{k_r}{2}. \tag{5}$$

For $G = O(V), n = 2r$, $k_{r-1} + k_r$ is even for $\chi \in P$, and we have:

$$n(\chi) = k_1 + 2k_2 + \cdots + (r-2)k_{r-2} + r\frac{k_{r-1} + k_r}{2} - \min(k_r, k_{r-1}). \tag{6}$$

These formulae yield the assertion of Lemma.

Consider the case $G = GL(V)$. The elements of $\mathcal{M}$ with non-zero weights have the following weights endowed with degrees:

$$\alpha_i = \varphi_i, \deg\alpha_i = i, i = 1, \cdots, n,$$

$$\beta_j = \varphi_j - \varphi_n, \deg\beta_j = n - j, j = 1, \cdots, n-1, \beta_n = -\varphi_n, \deg\beta_n = n.$$

For $\chi = k_1\varphi_1 + \cdots + k_n\varphi_n$ consider the presentations of χ as linear combinations of the above weights with positive integer coefficients. Define the degree of such a combination as the sum of degrees of the summands. We claim that there is a unique presentation of minimal degree.

For any $j = 1, \cdots, n-1$, all the presentations of χ contain k_j summands α_j or β_j. Set $r = [\frac{n}{2}]$. The linear combination

$$\chi' = k_1\alpha_1 + \cdots + k_r\alpha_r + k_{r+1}\beta_{r+1} + \cdots + k_{n-1}\beta_{n-1}$$

has the minimal degree among the linear combinations equal to χ modulo $\langle\varphi_n\rangle$. If $\chi' = \chi$, then this presentation of χ has the minimal degree and no presentation of the same degree exists. Otherwise, we can:

(a) replace some α_i by β_i, (b) add β_n,

(c) replace some β_j by α_j, (d) add α_n.

The steps (a),(b) decrease the n-th coordinate by 1, the steps (c),(d) increase it by 1. The increasing of the degree is: n for (b),(d), $n - 2i$ for (a), $2j - n$ for (c). If $\chi' = \chi + t\varphi_n$, then to obtain the minimal presentation, we apply t times (a) and (b), if $t > 0$, and we apply $-t$ times (c) and (d), if $t < 0$. Clearly, there exists a unique sequence of steps giving χ with the minimal possible degree. Therefore the presentation of χ with the minimal degree is unique. Moreover, from its construction follows that the presentation of $c\chi$ with the minimal degree is just the sum of c minimal presentations for χ. This completes the proof.□

Proof of Lemma 2.

Consider the case $G = GL(V)$. Let $\varepsilon_1, \cdots, \varepsilon_n$ be the weights of T acting on V, a basis of the character lattice of T. let $\chi_1, \cdots, \chi_n$ be the dual basis. The fundamental weights are: $\varphi_i = \varepsilon_1 + \cdots + \varepsilon_i, i = 1, \cdots, n$. Furthermore, $\mathcal{C}$ is given by the inequalities $\chi_1 \geq \chi_2 \cdots \geq \chi_n$, $\Phi = \operatorname{conv}(\pm\varepsilon_1, \cdots, \pm\varepsilon_n)$, and $\widetilde{\Delta}$ is the convex hull of

$$\varepsilon_1, \frac{\varepsilon_1 + \varepsilon_2}{2}, \cdots, \frac{\varepsilon_1 + \cdots + \varepsilon_n}{n}, -\varepsilon_n, \frac{-\varepsilon_n - \varepsilon_{n-1}}{2}, \cdots, \frac{-\varepsilon_n - \cdots - \varepsilon_1}{n}.$$

For $\chi \in \langle \varepsilon_1, \cdots, \varepsilon_n \rangle_{\mathbf{Q}}$, set $\alpha_i = \chi_i(\xi)$. First assume

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0, \alpha_1 + \cdots + \alpha_n \leq 1. \tag{7}$$

Then we can rewrite:

$$\xi = (\alpha_1 - \alpha_2)\varphi_1 + (\alpha_2 - \alpha_3)\varphi_2 + \cdots + (\alpha_{n-1} - \alpha_n)\varphi_{n-1} + \alpha_n\varphi_n.$$

So ξ is a linear combination of $\frac{\varphi_i}{i}, i = 1, \cdots, n$ with non-negative coefficients. Now we sum the coefficients:

$$(\alpha_1 - \alpha_2) + 2(\alpha_2 - \alpha_3) + \cdots + (n-1)(\alpha_{n-1} - \alpha_n) + n\alpha_n = \alpha_1 + \cdots + \alpha_n \leq 1.$$

Therefore we get:

$$\xi \in \operatorname{conv}(0, \varepsilon_1, \frac{\varepsilon_1 + \varepsilon_2}{2}, \cdots, \frac{\varepsilon_1 + \cdots + \varepsilon_n}{n}) \subseteq \widetilde{\Delta}.$$

Analogously, assuming

$$0 \geq \alpha_1 \geq \cdots \geq \alpha_n, \alpha_1 + \cdots + \alpha_n \geq -1, \tag{8}$$

we obtain

$$\xi \in \operatorname{conv}(0, -\varepsilon_n, \frac{-\varepsilon_n - \varepsilon_{n-1}}{2}, \cdots, \frac{-\varepsilon_n - \cdots - \varepsilon_1}{n}) \subseteq \widetilde{\Delta}.$$

Now assume $\xi \in \Phi \cap \mathcal{C}$. Then $\xi \in \Phi$ implies $|\alpha_1| + \cdots + |\alpha_n| \leq 1$. If all the α_i are of the same sign, then either (7) or (8) holds and we are done. Otherwise for some $q < n$ we have

$$\alpha_1 \geq \cdots \geq \alpha_q \geq 0 \geq \alpha_{q+1} \geq \cdots \geq \alpha_n.$$

Then set:

$$t = \sum_{i=1}^{q} \alpha_i \leq 1, \xi_+ = \frac{\sum_{i=1}^{q} \alpha_i \varepsilon_i}{t}, \xi_- = \frac{\sum_{j=q+1}^{n} \alpha_j \varepsilon_j}{1-t}.$$

Clearly, (7) holds for ξ_+ and (8) holds for ξ_-. Hence, $\xi_+, \xi_- \in \widetilde{\Delta}$, and $\xi = t\xi_+ + (1-t)\xi_- \in [\xi_+, \xi_-] \subseteq \widetilde{\Delta}$.

For $G = Sp(V), O(V)$, we let $\varepsilon_1, \cdots, \varepsilon_r$ to be basic characters of $T(G)$ and keep the notation of χ_i-s. Then the fundamental weights are (see e.g. [OV]):

for $G = Sp(V)$, $\varphi_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $i = 1, \cdots, r$,

for $G = O(V)$, $n = 2r + 1$, $\varphi_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $i = 1, \cdots, r-1$,
$\varphi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r)$,

for $G = O(V)$, $n = 2r$, $\varphi_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $i = 1, \cdots, r-2$,
$\varphi_{r-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r)$, $\varphi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r)$.

For the cases $G = Sp(V)$, $n = 2r$ or $G = SO(V)$, $n = 2r + 1$, we have: $\mathcal{C}$ is given by the inequalities $\chi_1 \geq \chi_2 \cdots \geq \chi_m \geq 0$,

$$\Phi = \mathrm{conv}(\pm\varepsilon_1, \cdots, \pm\varepsilon_r), \widetilde{\Delta} = \mathrm{conv}(0, \varepsilon_1, \frac{\varepsilon_1 + \varepsilon_2}{2}, \cdots, \frac{\varepsilon_1 + \cdots + \varepsilon_r}{r}).$$

Therefore for $\xi \in \mathcal{C} \cap \Phi$ the assumption (7) holds, hence $\xi \in \widetilde{\Delta}$.

For the case $G = O(V), n = 2r$, we have: $\Phi = \mathrm{conv}(\pm\varepsilon_1, \cdots, \pm\varepsilon_r)$,

$$\widetilde{\Delta} = \mathrm{conv}(0, \varepsilon_1, \frac{\varepsilon_1 + \varepsilon_2}{2}, \cdots, \frac{\varepsilon_1 + \cdots + \varepsilon_r}{r}, \frac{\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r}{r}).$$

If $\xi \in \mathcal{C}$, then we can write:

$$\xi = \alpha_1\varepsilon_1 + \alpha_2\frac{\varepsilon_1 + \varepsilon_2}{2} + \cdots + \alpha_r\frac{\varepsilon_1 + \cdots + \varepsilon_r}{r} + \beta\frac{\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r}{r},$$

where $\alpha_1, \cdots, \alpha_r, \beta \geq 0$, $\alpha_r\beta = 0$. Assume $\xi \in \Phi$. If $\alpha_r = 0$, then, taking into account the inequality $\chi_1(\xi) + \cdots + \chi_{r-1}(\xi) - \chi_r(\xi) \leq 1$, we obtain $\alpha_1 + \cdots + \alpha_{r-1} + \beta \leq 1$. Therefore $\xi \in \widetilde{\Delta}$. Similarly, we consider the case $\beta = 0$. This completes the proof of Lemma 2.$\square$

4 Syzygies.

Since we found the generators of $\mathbf{k}[W]^{U(G)}$, a natural question is to describe their syzygies. This is a subject of the Second Fundamental Theorem of Invariant Theory for the linear group $(U(G), V)$. In this section we present some results for $G = GL(V)$. Of course, syzygies that we present are also syzygies for the orthogonal and symplectic cases, if the involved generators are.

Set $U = U(GL)$ and denote by W_U the spectrum of $\mathbf{k}[W]^U$. Moreover, denote by $\pi_{U,W}$ the quotient map $\pi_{U,W} : W \to W_U$ corresponding to the inclusion $\mathbf{k}[W]^U \subseteq \mathbf{k}[W]$.

For any $p, l \in \mathbf{N}, 1 \leq p \leq l$, set $L = \mathbf{k}^l \oplus \wedge^2\mathbf{k}^l \oplus \cdots \oplus \wedge^p\mathbf{k}^l$. Let $\mathcal{F}_{p,l}$ denote the set of all $(q_1, q_2, \cdots, q_p)$ in L such that for $i = 2, \cdots, p$, the i-vector q_i is decomposable, and $Ann(q_{i-1}) \subseteq Ann(q_i)$, where $Ann(q) = \{x \in V | q \wedge x = 0\}$.

The subset $\mathcal{F}_{p,l}$ is not closed in L. In fact, assume $(q_1, \cdots, q_p) \in \mathcal{F}_{p,l}$ is such that $q_2 \neq 0$. Then for any $t \in \mathbf{k}^*$ the collection $(tq_1, q_2, \cdots, q_p)$ also belongs to $\mathcal{F}_{p,l}$. But the limit $(0, q_2, \cdots, q_p)$ of such collections does not belong to $\mathcal{F}_{p,l}$. Denote by $\overline{\mathcal{F}_{p,l}}$ the Zariski closure of $\mathcal{F}_{p,l}$,

Note that the subset $\mathcal{F}_{p,l}$ is stable under the natural action of the group GL_l on L. Therefore $\mathcal{F}_{p,l}$ is acted upon by GL_l.

Theorem 3 *For $W = lV$, set $p = \min\{l, n\}, q = n - p + 1$. Consider the rows $u_1, \cdots, u_n$ of the matrix $\overline{V}$ as the coordinates of some vectors in $\mathbf{k}^l$. Then the map $W \to \overline{\mathcal{F}_{p,l}} \subseteq L$ taking a tuple of vectors to the element with coordinates*

$$(u_n, u_{n-1} \wedge u_n, \cdots, u_q \wedge u_{q+1} \wedge \cdots \wedge u_n)$$

is the GL_L-equivariant quotient map $\pi_{U,W}$ and its image is $\mathcal{F}_{p,l}$.

Proof. We only need to prove that the Plücker coordinates of the antisymmetric forms $u_q \wedge \cdots \wedge u_n, \cdots, u_{n-1} \wedge u_n, u_n$ generate $\mathbf{k}[W]^U$. But these are just the lower minor determinants of $\overline{V}$ and Theorem 1 implies Theorem 3. A different proof of both Theorems for this case is as follows. Let the maximal unipotent subgroup $U' \subseteq GL_l$ consist of all the strictly upper triangular matrices, in the chosen basis of $\mathbf{k}^l$. It is well known (see e.g. [Kr, 3.7]) that $\mathbf{k}[W]^{U \times U'}$ is generated by the left lower minor determinants of $\overline{V}$. Therefore the algebra A generated by all the lower minor determinants contains $\mathbf{k}[W]^{U \times U'}$. In other words, $A^{U'} = (\mathbf{k}[W]^U)^{U'}$. Since A is GL_l-stable, we obtain $A = \mathbf{k}[W]^U$.□

Thus the syzygies of the set of lower minor determinants of $\overline{V}$ are the generators of the ideal in $\mathbf{k}[L]$ vanishing on $\mathcal{F}_{p,l}$. These are the Plücker relations saying that each q_i is decomposable, and the incidence relations saying $Ann(q_i) \subseteq Ann(q_j)$ for any $1 \leq i < j \leq p$.

The syzygies can be written down explicitly. For instance, if $i + j \leq p$, then we construct a $(i + j) \times l$ matrix of the last i rows and the last j rows of $\overline{V}$. Clearly, any minor determinant of order $i + j$ of such a matrix is zero. This is a bilinear syzygy among the lower minor determinants of order i and j.

There is also a GL_l-equivariant description of the ideal of syzygies, in the form of [Br85]. For $1 \leq i \leq j \leq p$, let $M_{i,j}$ be the GL_l-stable complementary subspace to the highest vector irreducible factor of $(\wedge^i \mathbf{k}^l)^* \otimes (\wedge^j \mathbf{k}^l)^* \subseteq \mathbf{k}[L]$, if $i < j$, or of $S^2(\wedge^i \mathbf{k}^l)^* \subseteq \mathbf{k}[L]$, if $j = i$. Let J be the ideal generated by $M_{i,j}$, for all $1 \leq i \leq j \leq p$.

Lemma 4 *The ideal in $\mathbf{k}[L]$ vanishing on $\mathcal{F}_{p,l}$ is J.*

Proof: Clearly, we have: $\overline{\mathcal{F}_{p,l}} = GL_l(L^{U'})$. Then by the Theorem of [Br85, p.382], the set of zeros of J is $\overline{\mathcal{F}_{p,l}}$. Moreover, by the same theorem, J is radical. This completes the proof.□

Corollary 1 *All the syzygies are of degree 2.*

Clearly, for arbitrary l and m, similar Plücker and incidence relations hold for the left minor determinants of $\overline{V^*}$.

Theorem 4 *Suppose that $l > 0, m > 0$ and set $W = lV + mV^*$. Then the ideal of syzygies for the generators of $\mathbf{k}[W]^U$ is generated by the Plücker and the incidence relations for the lower minor determinants of $\overline{V}$ and for the left minor determinants of $\overline{V^*}$ if and only if $l + m \leq n$.*

Proof: To prove the "if" part, it is sufficient to consider the case $l + m = n$. Recall that by Theorem 1, the generators of $\mathbf{k}[W]^U$ are the lower minor determinants of $\overline{V}$, the left minor determinants of $\overline{V^*}$, and the elements of the matrix $C = \overline{V^*V}$. Let $\sum_\alpha a_\alpha c^\alpha = 0$ be a relation among the generators, where c^α is a monomial in the C_i^j-s, a_α is a polynomial in the minor determinants. The assertion of the Theorem amounts to prove that a_α belongs to the ideal of syzygies, for any α. This will be proven if we check for generic fibers $F = \pi^{-1}_{U,lV}(\xi), \xi \in \mathcal{F}_{l,l}$ and $F^* = \pi^{-1}_{U,mV^*}(\eta), \eta \in \mathcal{F}_{m,m}$ that the restrictions of the matrix elements of C to $F \times F^*$ are algebraically independent. Fix a tuple of vectors in a generic fiber F such that $\overline{V}$ has the form

$$\left(\begin{array}{ccc} \leftarrow & l & \rightarrow \\ * & * & * \\ * & * & * \\ \hline a_1 & 0 & 0 \\ * & \ddots & 0 \\ * & * & a_l \end{array}\right)$$

with $a_1 a_2 \cdots a_l \neq 0$ and fix generic elements of the first m columns of $\overline{V^*}$. Then, varying the lm elements in the last $l = n - m$ columns of $\overline{V^*}$, we do not change the minor determinants and we can obtain any $m \times l$ matrix as C. Thus the "if" part is proven.

The "only if" part. Take l, m such that $1 \leq l, m \leq n, l + m > n$ and set $s = l + m - n, r = n - l + 1$. Denote by a_i^j, b_i^j, c_i^j the element in the i-th row and the j-th column of the matrix $\overline{V^*}, \overline{V}$, and C, respectively. Denote by $\varepsilon^{a\cdots b}$ and $\varepsilon_{a\cdots b}$ the determinant tensors. In this notation, $\varepsilon^{i_1\cdots i_m} a^1_{i_1} \cdots a^m_{i_m}$ is the left minor determinant of order m of $\overline{V^*}$ and $\varepsilon_{j_1\cdots j_l} b^{j_1}_r \cdots b^{j_l}_n$ is the lower minor determinant of order l of $\overline{V}$. We claim that the following relation holds[2]:

$$\varepsilon^{i_1\cdots i_m} a^1_{i_1} \cdots a^m_{i_m} \varepsilon_{j_1\cdots j_l} b^{j_1}_r \cdots b^{j_l}_n = \tag{9}$$

$$= \frac{1}{s!} \varepsilon^{i_1\cdots i_m} a^1_{i_1} \cdots a^{r-1}_{i_{r-1}} \varepsilon_{j_1\cdots j_l} b^{j_{s+1}}_{m+1} \cdots b^{j_l}_n c^{j_1}_{i_r} \cdots c^{j_s}_{i_m}.$$

To prove this formula, we rewrite the right hand side, using $c_i^j = a_i^k b_k^j$:

$$\frac{1}{s!} \varepsilon^{i_1\cdots i_m} a^1_{i_1} \cdots a^{r-1}_{i_{r-1}} a^{k_1}_{i_r} \cdots a^{k_s}_{i_m} \varepsilon_{j_1\cdots j_l} b^{j_1}_{k_1} \cdots b^{j_s}_{k_s} b^{j_{s+1}}_{m+1} \cdots b^{j_l}_n. \tag{10}$$

[2]This relation with $m = n$ was indicated to us by E. B. Vinberg.

Let $S(k_1, \cdots, k_s)$ denote the sum of terms in formula (10) with fixed $k_1, \cdots, k_s$. Clearly, if $\{k_1, \cdots, k_s\} \neq \{r, r+1, \cdots, m\}$, then $S(k_1, \cdots, k_s) = 0$. Moreover, if $\{k_1, \cdots, k_s\} = \{r, \cdots, m\}$, then $S(k_1, \cdots, k_s)$ equals the left hand side of (9).

Therefore, the relation (9) holds. Clearly, the right hand side is a polynomial in the left minor determinants of order $m-s$ of $\overline{V^*}$, the lower minor determinants of order $l-s$ of $\overline{V}$, and the matrix elements of C. It is not hard to check that this relation among the generators of $\mathbf{k}[W]^U$ can not be obtained from relations of smaller degrees.$\square$

Remark. Theorems 3, 4 yield an independent proof of Theorem 1 for the case $l+m \leq n$. Indeed, we prove in Theorem 4 that, in the case $l+m \leq n$, the syzygies among the elements of the set $\mathcal{M}$ are generated by those for lV and those for mV^*. We did not use Theorem 1 for this. Hence, by Theorem 3 (that we also prove independently of Theorem 1), Spec$A \cong (lV)_U \times (mV^*)_U$. Since for an action of an algebraic group H on a normal affine variety X, the algebra $\mathbf{k}[X]^H$ is integrally closed, Spec$A$ is normal. Furthermore, as we did it for $O(V)$, one can prove $\mathbf{k}[W]^U_f = A_f$ for all linear U-invariants f. Then any $g \in \mathbf{k}[W]^U$ gives rise to a rational function on Spec$A$, regular outside the intersection of the divisors of these linear U-invariants. Since Spec$A$ is normal, we get $g \in A$.

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